Integral online class

1er novembre 2017

Definition 0.1. *If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 (= a), x_1, x_2, \cdots, x_n (= b)$ be the endpoints of these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then, the* **definite integral of f from a to b** *is*

$$\int_a^b f(x)dx = lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists. if it does exist, we say that f is **integrable** *on $[a, b]$.*

Note that the symbol $\int$ was introduced by Leibniz and is called an **integral sign**. *$f(x)$ is called the* **integrand** *and a and b are called the* **limits of integration** *; a is the* **lower limit** *and b is the* **upper limit**. *The dx simply indicates that the dependent variable is x. $\int_a^b f(x)dx$ is all one symbol. The procedure of calculating an integral is called* **integration**. *We can write $\int f(x)dx$ for the antiderivative of f when x is chosen to be the variable.*

Note that

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

If $f(x) \geqslant 0$, the integral $\int_a^b f(x)dx$ is the area delimited by the graph of f, the x-axis and the line $x = a$ and $x = b$.

When $f(x) \leqslant 0$ the integral $\int_a^b f(x)dx$ is minus the area delimited by the graph of f, the x-axis and the line $x = a$ and $x = b$.

In general, $\int_a^b f(x)dx$ is the sum of the areas above the x-axis minus the sum of the areas bellow the x-axis (for the domains delimited by the graph of f, the x-axis and the line of equation $x = a$ and $x = b$).

Theorem 0.2. *If f is continuous on $[a, b]$, or f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is the definite integral $\int_a^b f(x)dx$ exists.*

Theorem 0.3. *If f is integrable on $[a, b]$ then*

$$\int_a^b f(x)dx = lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where

$$\Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x$$

Example 0.4. *1. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b - 3$ and $n = 6$.*

2. Evaluate

$$\int_0^3 (x^3 - 6x)dx$$

Solution :

1. *With $n = 6$ the interval width is*

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

and the right endpoints are $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$ *and* $x_6 = 3.0$. *So the Riemann sum is*

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = f(0.5)1/2 + f(1.0)1/2 + f(1.5)1/2 + f(2)1/2 + f(2.5)1/2 + f(3)1/2 = -3.9375$$

2. *With n subintervals we have*

$$\Delta x = \frac{b-a}{n} = 3/n$$

Thus $x_0 = 0, x_1 = 3/n, x_2 = 6/n, x_3 = 9/n$, *and in general* $x_i = 3i/n$. *Since we are using endpoints, we can use the previous theorem* 4 *:*

$$\begin{array}{rcl} \int_0^3 (x^3 - 6x)dx & = & lim_{n\to\infty} f(x_i)\Delta x \\ & = & lim_{n\to\infty} f(3i/n)3/n \\ & = & lim_{n\to\infty} \left[(3i/n)^3 - 6(3i/n)\right] 3/n \\ & = & lim_{n\to\infty} \left[81/n^4 \sum_{i=1}^{n} i^3 - 54/n^2 \sum_{i=1}^{n} i\right] \\ & = & lim_{n\to\infty} \left[81/n^4 \left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right] \\ & = & lim_{n\to\infty} [\frac{81}{4}(1+1/n)^2 - 27(1+1/n)] \\ & = & 81/4 - 27 = -6.75 \end{array}$$

Example 0.5. *Evaluate the following integrals by interpreting each in terms of areas*

1. $\int_0^1 \sqrt{1-x^2}dx$

Solution : *Since* $f(x) = \sqrt{1-x^2} \geqslant 0$, *we can interpret this integral as the area delimited by the graph of* f *the* x*-axis and the line of equation* $x = 0$ *and* $x = 1$. *But since* $y^2 = 1 - x^2$, *we get* $x^2 + y^2 = 1$, *which shows that the graph of* f *is the quarte-circle with radius* 1 *see figure bellow :*

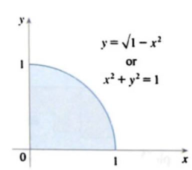


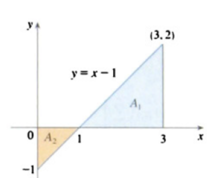
FIGURE 9

Therefore,

$$\int_0^1 \sqrt{1-x^2}dx = 1/4\pi 1^2 = \pi/4$$

2. $\int_0^3 (x-1)dx$.

Solution : *The graph of* $y = x - 1$ *is the line with slope* 1 *in the figure bellow :*



We compute the integral as a difference of the area of the two triangles :

$$\int_0^3 (x-1)dx = A_1 - A_2 = 1/2(2 \cdot 2) - 1/2(1 \cdot 1) = 1.5$$

Properties of the definite integral

Theorem 0.6.

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b cdx = c(b-a)$, *where* c *is any constant.*
4. $\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ *(We will refer to this property as linearity property of the integral)*
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$, *where* c *is any constant. (We will also refer to this property as linearity property of the integral)*
6. $\int_a^b (f(x)-g(x))dx = \in_a^b f(x)dx - \int_a^b g(x)dx$. *(We will also refer to this property as linearity property of the integral)*
7. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ *(We will refer to this property as additive property of the integral)*
8. *If* $f(x) \geqslant 0$ *for* $a \leqslant x \leqslant b$, *then* $\int_a^b f(x)dx \geqslant 0$ *(We will refer to this property as comparison property of the integral)*
9. *If* $f(x) \geqslant g(x)$, *for* $a \leqslant x \leqslant b$, *then* $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$. *(We will refer to this property as comparison property of the integral)*
10. *If* $m \leqslant f(x) \leqslant M$ *for* $a \leqslant x \leqslant b$, *then*

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

(We will refer to this property as comparison property of the integral)

Exercise : Try to prove those using the definition of the integral.

Example 0.7. *Use the properties of integrals to evaluate* $\int_0^1 (4+3x^2)dx$.
Solution : *Using the linearity property of the integral we get*

$$\int_0^1 (4+3x^2)dx = \int_0^1 4dx + \int_0^1 3x^2dx = \int_0^1 4dx + 3\int_0^1 x^2dx$$

We have from the previous properties of the integral that

$$\int_0^1 4dx = 4(1-0) = 4$$

and we have proven in the previous section that

$$\int_0^1 x^2dx = 1/3$$

So,

$$\int_0^1 (4+3x^2)dx = 4 + 3 \cdot 1/3 = 5$$

Example 0.8. *If it is known that* $\int_0^{10} f(x)dx = 17$ *and* $\int_0^8 f(x)dx = 12$*, find* $\int_8^{10} f(x)dx$.

Solution : *By the additive property of the integral we have that*

$$\int_0^8 f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx$$

so

$$\int_8^{10} f(x)dx = \int_0^{10} f(x)dx - \int_0^8 f(x)dx = 17 - 12 = 5$$

Example 0.9. *Estimate* $\int_0^1 e^{-x^2}dx$ *using comparison properties.*

Solution : *Because* $f(x) = e^{-x^2}$ *is a decreasing function on* $[0, 1]$*, its absolute maximum value is* $M = f(0) = 1$ *and its absolute minimum value is* $m = f(1) = e^{-1}$*. Thus by the comparison properties we know that*

$$e^{-1}(1-0) \leqslant \int_0^1 e^{-x^2}dx \leqslant 1(1-0)$$

So that

$$e^{-1} \leqslant \int_0^1 e^{-x^2}dx \leqslant 1$$

Note that $e^{-1} \approx 0.3679$.